N-Cluster Search: Progress Update

Authors

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Abstract

We recently found 25 7-clusters using 25,000 cores over 12 days. This work was an extension to our discovered of the first 7-cluster back in 2006-May-18. We present a method for constructing 7-clusters that are not necessarily the smallest 7-cluster.

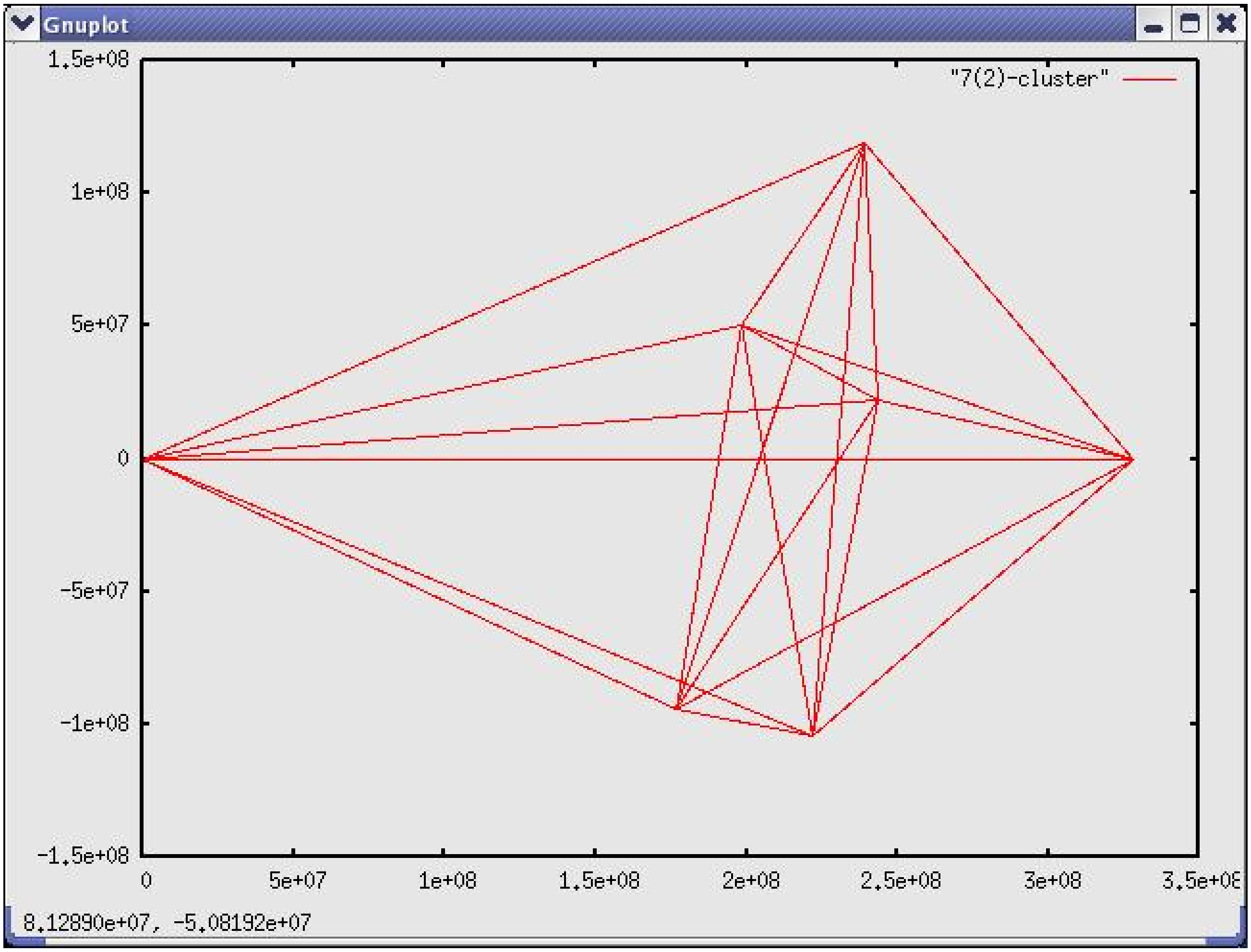
Introduction

In a problem made famous by Paul Erdös: Problem D20 of *Unsolved Problems in Number Theory, 2nd Edition* [1] asks what is the largest set of points in a plane such that the points are pairwise a rational distance apart, no three collinear, no four cocircular.

Noll and Bell defined an “N-cluster” to be a set of points in a plane with integer coordinates such that the points are an integer distance apart, no three collinear, no four cocircular. 6-clusters were found in 1989-July by Noll & Bell [2] and independently discovered by Rathbun [4].

Proof of the existence of 7-clusters in R^2

We discovered first 7-cluster in R^2 on 2006-May-18 15:26:55 PDT [5]:



(0,0) (327990000,0) (238776720,118951040) (222246024,-103907232)
(243360000,21896875) (198368352,50379264) (176610000,-94192000)

The algorithm used to discover the above 7-cluster was found using calc [3].

Constructing N-Clusters

Any three points of an N-cluster form the vertices of a Heron triangle. Each Heron triangle can be constructed from a pair of Pythagorean triangles. Pythagorean triangles can be generated by choosing relatively prime integers *n, m* with *m* > *n.* The sides of the Pythagorean triangle are then *m^2* + *n^2, 2mn*, and *m^2* - *n^2*.

The angles of a Pythagorean triangle are Pythagorean angles. We can construct a Heron triangle by choosing any two Pythagorean angles that sum to less than 180 degrees.

We can construct a 4-cluster by choosing a pair of Heron triangles, an orientation for the first triangle, and a relative orientation for the second triangle. We dilate the triangles to have a common base length and glue the triangles together at their bases. We then test to see if the resulting construct meets integer distance, colinearity, and cocircularity constraints, discarding those constructs that fail one or more of these tests.

Given a 3-cluster, we can construct a list of 4-clusters that contain dilated copies of the three cluster. Given a list of N-clusters that contain dilated variants of a common (N-1)-cluster, we can construct a list of (N+1) clusters by choosing two N-clusters, dilating them to have the same sized (N-1)-cluster, gluing them together on the common (N-1)-cluster, and checking the result to see if it meets integer distance, colinearity, and cocircularity constraints.

If we use *k* bits to store the length of an N-cluster, we need up to *2k+1* bits to store the length of an (N+1)-cluster constructed from a pair of intersectable N-clusters. Thus if we use 7-bits to hold the *n,m* seeds for a Pythagorean triangle, we need 15-bits for the sides of the Pythagorean triangle, 31 bits for the sides of a Heron triangle, 63 bits for the sides of a 4-cluster, etc. These numbers, of course, are the number of bits used in common signed integer datatypes in modern programming languages. [We used the BigNum library to handle multi-precision arithmetic in C++.]

The Search

We performed a search in 2010-Aug restricting the Pythagorean seeds *n, m* to be less than 128. These seeds generate a list of around 40 million Heron triangles. We parallelized the search and then ran the search across 25,000 cpu cores at Google over 12 days. We found 25 distinct 7-clusters during the search, up from the three we had previously found in 2006-May.

During the search, we tested:

- 3 797 046 086 162 688 (3.8 quintillion) pairs of 3-clusters. This generated at least one list of 29 208 4-clusters that shared a common 3-cluster. (This was probably a list of mostly isosceles triangles.)

- 224 035 314 871 pairs of 4-clusters. (The longest list of 5-clusters sharing a common 4-cluster contained 64 elements.)

- 3 146 308 pairs of 5-clusters. (The longest list of 6-clusters sharing a common 5-cluster contained 4 elements.)

- 532 pairs of 6-clusters. (The longest list of 7-clusters sharing a common 6-cluster was 1; we did not find any pairs of intersectable 7-clusters to test.)

The Results

In addition to the previously mentioned first known 7-cluster, we present the following 7-clusters in R^2:

(0,0) (327990000,0) (238776720,118951040) (222246024,-103907232) (243360000,21896875) (198368352,50379264) (176610000,-94192000)

(68634995347500,46759948729375) (85941805950000,71210832840000) (68634995347500,4632341698125) (130072970722500,48916330870000) (133822723450416,98216877284288) (137269990695000,0) (0,0)

(68634995347500,46759948729375) (51328184745000,71210832840000) (68634995347500,4632341698125) (7197019972500,48916330870000) (3447267244584,98216877284288) (0,0) (137269990695000,0)

(8742402935396,-10550893265055) (12491072799192,-7329880708800) (1251972023796,-12174625385247) (8742402935396,590045900847) (17484805870792,0) (0,0) (5153771043792,-23774836660350)

(8742402935396,-10550893265055) (4993733071600,-7329880708800) (16232833846996,-12174625385247) (8742402935396,590045900847) (0,0) (17484805870792,0) (12331034827000,-23774836660350)

(57074946589137,-29602439091140) (68119373772450,-49995038226600) (57074946589137,6361294356684) (108165052231587,-40677455540084) (96768547697250,-80181802070200) (114149893178274,0) (0,0)

(57074946589137,-29602439091140) (46030519405824,-49995038226600) (57074946589137,6361294356684) (5984840946687,-40677455540084) (17381345481024,-80181802070200) (0,0) (114149893178274,0)

(127760562354100,174076664586825) (189715358500200,126876727286400) (18296188244100,177918702348075) (127760562354100,14239567321200) (255521124708200,0) (0,0) (12519023151384,356682343821888)

(127760562354100,174076664586825) (65805766208000,126876727286400) (237224936464100,177918702348075) (127760562354100,14239567321200) (0,0) (255521124708200,0) (243002101556816,356682343821888)

(1481541457500,-537780709375) (1665923415000,0) (832961707500,-1005271678125) (832961707500,432023150000) (1550362069464,1036842072448) (0,0) (1523288063400,-2113353748800)

(184381957500,-537780709375) (0,0) (832961707500,-1005271678125) (832961707500,432023150000) (115561345536,1036842072448) (1665923415000,0) (142635351600,-2113353748800)

(5272996965444,-1914030845545) (5929236112488,0) (2964618056244,-4039359357033) (2964618056244,2019747908633) (6173748624888,3622814603200) (0,0) (6745851981888,-7326859050450)

(5272996965444,-1914030845545) (5929236112488,0) (2964618056244,-4039359357033) (2964618056244,2019747908633) (6173748624888,3622814603200) (0,0) (6745851981888,-7326859050450)

(656239147044,-1914030845545) (0,0) (2964618056244,-4039359357033) (2964618056244,2019747908633) (-244512512400,3622814603200) (5929236112488,0) (-816615869400,-7326859050450)

(1481541457500,-537780709375) (1665923415000,0) (832961707500,-1005271678125) (832961707500,432023150000) (1550362069464,1036842072448) (0,0) (1523288063400,-2113353748800)

(184381957500,-537780709375) (0,0) (832961707500,-1005271678125) (832961707500,432023150000) (115561345536,1036842072448) (1665923415000,0) (142635351600,-2113353748800)

(68634995347500,46759948729375) (85941805950000,71210832840000) (68634995347500,4632341698125) (130072970722500,48916330870000) (133822723450416,98216877284288) (137269990695000,0) (0,0)

(68634995347500,46759948729375) (51328184745000,71210832840000) (68634995347500,4632341698125) (7197019972500,48916330870000) (3447267244584,98216877284288) (0,0) (137269990695000,0)

(8742402935396,-10550893265055) (12491072799192,-7329880708800) (1251972023796,-12174625385247) (8742402935396,590045900847) (17484805870792,0) (0,0) (5153771043792,-23774836660350)

(8742402935396,-10550893265055) (4993733071600,-7329880708800) (16232833846996,-12174625385247)

(8742402935396,590045900847) (0,0) (17484805870792,0) (12331034827000,-23774836660350)

(57074946589137,-29602439091140) (68119373772450,-49995038226600) (57074946589137,6361294356684) (108165052231587,-40677455540084) (96768547697250,-80181802070200) (114149893178274,0) (0,0)

(57074946589137,-29602439091140) (46030519405824,-49995038226600) (57074946589137,6361294356684) (5984840946687,-40677455540084) (17381345481024,-80181802070200) (0,0) (114149893178274,0)

(127760562354100,174076664586825) (189715358500200,126876727286400) (18296188244100,177918702348075) (127760562354100,14239567321200) (255521124708200,0) (0,0) (12519023151384,356682343821888)

(127760562354100,174076664586825) (65805766208000,126876727286400) (237224936464100,177918702348075) (127760562354100,14239567321200) (0,0) (255521124708200,0) (243002101556816,356682343821888)

(214219880550,6301947600) (224858052864,-26545742352) (175084785954,0) (121350268260,-110224651680) (134680604580,-179574139440) (0,0) (345929626949,-494479514400)

(617451851250,-875785365000) (780757128000,-819794984400) (525254357862,-700339143816) (804949411500,-154727118000) (1122338371500,0) (0,0) (3015706938447,99720035404)

(1009529065706589428,0) (1203398391990949336,45860386955466720) (0,0) (676939921408158408,985583027337548640) (53761902907451508,-563219935220920560) (1246841707833420832,1354229751267244425) (2606346791263681648,-1834662143896632960)

(220244176831120,0) (246555468595800,12985596405600) (165138740488800,35797817293056) (187901542064600,73409269700325) (242381054588400,120746605948800) (95836927172560,134624270161920) (0,0)

(1541709237817840,0) (1725888280170600,90899174839200) (1155971183421600,250584721051392) (1315310794452200,513864887902275) (1982197591480080,1057172048789376) (670858490207920,942369891133440) (0,0)

(0,0) (-1203015538725,-2544411343500) (10749021956400,0) (19954499237400,4909587883200) (11374103163150,-17184050629200) (23379026610072,-4158083198304) (24966478323150,2041480914200)

(1163564931724079313,-269477403542383960) (834417531683024619,0) (1455548309962875744,134645123549696760) (270089852186798469,-710440427520830440) (945507656505731319,829472932009543360) (0,0) (540705461390553459,-3959673836535537120)

(2839035903168,420142221576) (2716494421875,0) (3488742000000,411865375000) (678132000000,-760880250000) (1889735250000,-1984222012500) (3053812164000,-2179591564500) (0,0)

(888913935,1185218580) (732147500,1624350000) (750638434,291746112) (0,0) (1899388750,0) (1508338125,-938521500) (3000817152,354263136)

(90559954461600,-56348416109440) (66832836431400,-65671968304800) (99758004738288,0) (62274616513200,7351864449475) (0,0) (76736926721760,102315902295680) (219297144627600,85232898316800)

(645746712,1432660320) (1133937120,1511916160) (920228400,108638075) (0,0) (1474118256,0) (1338199200,-832657280) (3240541200,1259481600)

(46093983313228774560,7133592655618738920) (51205858532566473645,14766053479192164552) (30836021244266471870,0) (37713496340872872000,-14317464299263259160) (48652889618418976800,-16680990726315077760) (23093827308239198310,-11696944424489686080) (0,0)

(204525733500,0) (181222195380,31071384160) (116714131380,-89881235840) (64902832764,110138140448) (107446130880,167222282535) (0,0) (195966456000,492121242000)

(1537489668,2659073040) (720014022,3200062320) (0,0) (3448272282,0) (-1907771943,1360265200) (8837240832,1733306400) (970646382,-5946302160)

(1861671240,2343685344) (1416176720,1989335040) (2776610200,1084764525) (3581650800,1784265600) (2440245600,528982272) (3254535440,0) (0,0)

(13031698680,16405797408) (9913237040,13925345280) (19436271400,7593351675) (17081719200,3702875904) (29290818960,15621770112) (22781748080,0) (0,0)

(18880367688,-8041182240) (19189941309,-15711063060) (20689633692,0) (28850942484,-6681773280) (26355138250,5713113840) (37802741526,-10726662720) (0,0)

(5722762500,7204470000) (6540315210,9187326720) (6160119966,3068777712) (8604820224,3361726368) (10778796875,2336565000) (8461703250,0) (0,0)

(5722762500,7204470000) (6540315210,9187326720) (5123868750,2732730000) (8604820224,3361726368) (10778796875,2336565000) (8461703250,0) (0,0)

(154949507036870777,-230980631607757680) (69052004404618851,-181633390270652760) (91144391325948576,-327981711210799800) (176997658235793101,0) (-102311826723971249,-274457955444334200) (0,0) (906675315979444011,209982703452642120)

(4959471309432841342089,0) (5145703589278133161956,-1817658122369063017680) (2728622318116179176640,819077651069860662480) (3631649471438236223040,2723737103578677167280) (5383609759475803626444,3592702165069798173360) (0,0) (50220507702021572864,1908048894599832784800)

(335600575349000,223959939060000) (460196641548696,345147481161522) (526540779804000,0) (451186570951000,-159376248780000) (0,0) (877918993526536,263533660291902) (35997201279000,1367656824909375)

(220540073481,238652060192) (164939775000,277533828000) (270450480300,0) (0,0) (-140026136250,336062727000) (299378956800,679511448000) (760141882500,493806456000)

(3749181249177,4057085023264) (2803976175000,4718075076000) (4597658165100,0) (0,0) (-1487477641650,8113514409000) (5089442265600,11551694616000) (12922412002500,8394709752000)

(21945462903,10830898260) (21142072098,0) (37836432900,12972491280) (0,0) (8666936388,38519717280) (54627962688,-22761651120) (60508055894,68284953360)

(122766861254427,47715055386336) (139881033166875,0) (0,0) (100323715366875,205792236650000) (274032673366875,100613730150000) (-22482696967185,190441668427920) (-38698090055625,-377699815350000)

(90026022240,262575898200) (0,0) (473881430880,419911601252) (-110715848880,631802398500) (427936195995,0) (680445131520,779676713200) (1105723338720,460718057800)

Pictures and coordinates of 7-clusters listed can be found on a blog posting by the 1st co-author [6] and a N-cluster status page maintained by the 2nd co-author [7].

Our 2010-Aug seach did not discover any 8-clusters in R^2.

It should be noted that Tobias Kreisel and Sascha Kurz independently reported the discovery of several 7-clusters in R^2 in a paper dated 2006-Nov-7 [8].

References

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[6] Simmons, Chuck and Noll, Landon Curt; “**Pictures and coordinates of twenty-five recently found 7-clusters**”, http://cesium062.blogspot.com/2010/10/pictures-and-coordinates-of-twenty-five.html , 2012-Oct-22.

[7] Noll, Landon Curt, Rathbun, Randall, and Simmons, Chuck; “**n_m-clusters**”, http://www.isthe.com/chongo/tech/math/n-cluster/index.html , 1999-2012.

[8] Kreisel, T. and Kurz, S.; "**There are Integral Heptagons, No Three Points on a Line, No Four on a Circle.**", http://www.wm.uni-bayreuth.de/fileadmin/Sascha/Publikationen2/rare.pdf and http://arxiv.org/abs/0804.1303, 2006-Nov-7.